The Determination of the Conjugate Points for Discontinuous Solutions in the Calculus of Variations.

By Oskar Bolza.

In §§ 8 and 9 of his Inaugural-Dissertation, "*Ueber die discontinuierlichen Lösungen in der Variationsrechnung*" (Göttingen, 1904), Caratheodory develops the general theory of the conjugate points for discontinuous solutions. The object of the present note is to derive Caratheodory's results concerning conjugate points by a more direct method, to supplement them in certain points, and to give in particular, in explicit form, the equation which connects the parameters of a pair of conjugate points.

§ 1. *Sets of "Broken Extremals".*

In order that a curve $P_1 P_0 P_2$ with a "corner" at P_0, but otherwise of class* C', may minimize† the integral

$$J = \int_{t_1}^{t_2} F(x, y, x', y')\, dt,$$

it is in the first place necessary that the two "continuous" branches $P_1 P_0$ and $P_0 P_2$ should separately satisfy the four necessary conditions for a minimum with fixed endpoints. In particular, each one of the two arcs $P_1 P_0$ and $P_0 P_2$ must be an *extremal.*

Further, at the point $P_0\,(x_0, y_0)$ Weierstrass-Erdmann's *corner-condition* ‡ must be satisfied:

$$\begin{aligned} F_{x'}(x_0, y_0, \cos\vartheta_0, \sin\vartheta_0) &= F_{x'}(x_0, y_0, \cos\bar{\vartheta}_0, \sin\vartheta_0), \\ F_{y'}(x_0, y_0, \cos\vartheta_0, \sin\vartheta_0) &= F_{y'}(x_0, y_0, \cos\bar{\vartheta}_0, \sin\bar{\vartheta}_0), \end{aligned} \tag{1}$$

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* Compare for the terminology my *Lectures on the Calculus of Variations*, § 2, c) and § 24, a).

† In the sense defined in § 24, c) of my *Lectures* and under the assumptions concerning the function $F(x, y, x', y')$ stated in § 24, b).

‡ Compare *Lectures*, § 25, c).

where ϑ_0 denotes the amplitude of the positive tangent to the arc $P_1 P_0$ at P_0, $\bar{\vartheta}_0$ the amplitude of the positive tangent to the arc $P_0 P_2$ at P_0.

We shall call a curve $P_1 P_0 P_2$ consisting of two arcs of extremals $P_1 P_0$ and $P_0 P_2$ a "*broken extremal*", if at P_0 this corner-condition (1) is satisfied.

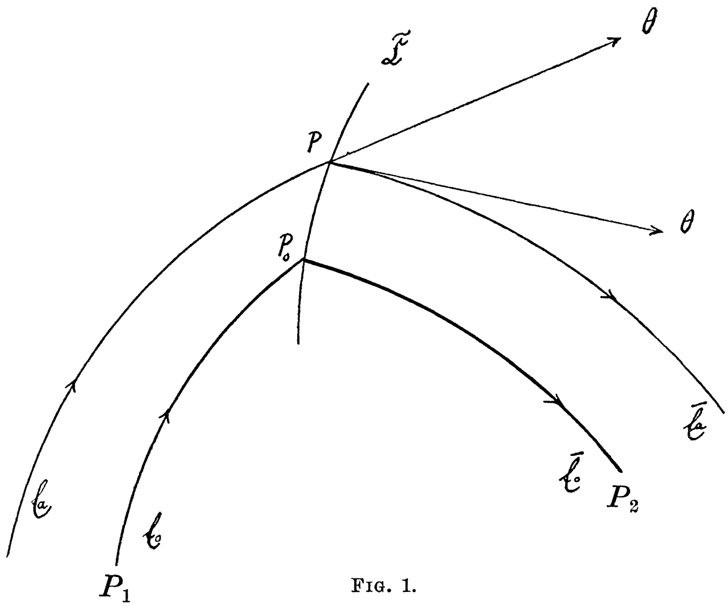

FIG. 1.

We assume for the following discussion that the curve $P_1 P_0 P_2$ lies in the *interior* of the domain of continuity R of the function F (compare *Lectures*, § 24, b), and that Legendre's condition is satisfied in the stronger form*

$$F_1 > 0 \tag{2}$$

along each of the two branches $P_1 P_0$ and $P_0 P_2$.

Let now

$$x = \phi(t, a), \quad y = \psi(t, a) \tag{3}$$

be any one-parameter set of extremals which contains the arc $P_1 P_0$ for $a = a_0$, so that the arc $P_1 P_0$ is representable by the equations

$$x = \phi(t, a_0), \quad y = \psi(t, a_0), \quad t_1 \leqq t \leqq t_0. \tag{4}$$

The functions

$$\phi, \ \phi_t, \ \phi_{tt}; \quad \psi, \ \psi_t, \ \psi_{tt}$$

* Compare *Lectures*, § 27, b).

are supposed * to be of class C' as functions of t and a in the domain

$$t_1 - h_1 \overline{\overline{<}} t \overline{\overline{<}} t_0 + h_0, \quad | a - a_0 | \overline{\overline{<}} d,$$

h_0, h_1, d being sufficiently small positive quantities.

The extremal of the set (3) corresponding to a particular value a will be denoted by $\mathfrak{E}_a$; further we write $\mathfrak{E}_0$ for $\mathfrak{E}_{a_0}$.

We propose to determine a point $P(t)$ on a given extremal $\mathfrak{E}_a$ of the set (3), and at the same time a direction $\bar{\vartheta}$ passing through P, such that the direction $\bar{\vartheta}$ together with the direction ϑ of the positive tangent to the extremal $\mathfrak{E}_a$ at P shall satisfy Weierstrass-Erdmann's corner-condition for the point P.

We have, then, for the determination of the two unknown quantities t and $\bar{\vartheta}$, the two equations †:

$$\begin{aligned} F_{x'}[\phi(t,a), \psi(t,a), \phi_t(t,a), \psi_t(t,a)] - F_{x'}[\phi(t,a), \psi(t,a), \cos\bar{\vartheta}, \sin\bar{\vartheta}] = 0, \\ F_{y'}[\phi(t,a), \psi(t,a), \phi_t(t,a), \psi_t(t,a)] - F_{y'}[\phi(t,a), \psi(t,a), \cos\bar{\vartheta}, \sin\bar{\vartheta}] = 0. \end{aligned} \tag{5}$$

These equations are satisfied for $t = t_0$, $a = a_0$, $\bar{\vartheta} = \bar{\vartheta}_0$, since according to our assumptions the broken extremal $P_1 P_0 P_2$ satisfies the corner-condition (1). Further, their left-hand members, which we denote by $\Phi(t, a, \bar{\vartheta})$ and $\Psi(t, a, \bar{\vartheta})$ respectively, are of class C' in the vicinity of the point $t_0, a_0, \bar{\vartheta}_0$. Hence we can apply the theorem on implicit functions,‡ provided that the Jacobian

$$J_{t\bar{\vartheta}} = \frac{\partial(\Phi, \Psi)}{\partial(t, \bar{\vartheta})}$$

is different from zero at the point $t_0, a_0, \bar{\vartheta}_0$. If we write for brevity

$$\cos\vartheta = p, \ \sin\vartheta = q; \ \cos\bar{\vartheta} = \bar{p}, \ \sin\bar{\vartheta} = \bar{q},$$

and remember that along the extremal $P_1 P_0$

$$\frac{\partial}{\partial t} F_{x'} = F_x, \quad \frac{\partial F_{y'}}{\partial t} = F_y,$$

we obtain:

$$\begin{aligned} \Phi_t &= F_x - \bar{F}_{x'x}\, x' - \bar{F}_{x'y}\, y', & \Psi_t &= F_y - \bar{F}_{y'x}\, x' - \bar{F}_{y'y}\, y', \\ \Phi_{\bar{\vartheta}} &= \bar{F}_1 \bar{q}, & \Psi_{\bar{\vartheta}} &= -\bar{F}_1 \bar{p}, \end{aligned} \tag{6}$$

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* The existence of an infinitude of sets of extremals satisfying these conditions is a consequence of our assumptions according to certain existence theorems on differential equations; compare Kneser, *Lehrbuch der Variationsrechnung*, § 27, and Bolza, *Trans. Amer. Math. Soc*, Vol. VII (1906), p. 464.

† Since $F_{x'}$, $F_{y'}$ are positively homogeneous of dimension zero in x', y', we may replace in these functions $\cos\vartheta$, $\sin\vartheta$ by $\phi_t(t, a)$, $\psi_t(t, a)$.

‡ Compare, for instance, Osgood, *Lehrbuch der Functionentheorie*, Vol. I, p. 52.

where the arguments of F_x, F_y are: $\phi(t, a)$, $\psi(t, a)$, $x' = \phi_t(t, a)$, $y' = \psi_t(t, a)$; those of $\bar{F}_1$, $\bar{F}_{x'x}$, etc.: $\phi(t, a)$, $\psi(t, a)$, $\bar{p}$, $\bar{q}$.

Making use of the homogeneity properties* of the function F and its partial derivatives, we obtain for the above Jacobian:

$$J_{t\bar{\vartheta}} = \sqrt{x'^2 + y'^2}\, \bar{F}_1 \{ p \bar{F}_x + q \bar{F}_y - (\bar{p} F_x + \bar{q} F_y) \}, \tag{7}$$

where now the two last arguments in F_x, F_y are p, q.

The first two factors of $J_{t\bar{\vartheta}}$ are different from zero for $t = t_0$, $a = a_0$, $\bar{\vartheta} = \vartheta_0$. Hence if we put, with Caratheodory,

$$\begin{aligned} \Omega(x_0, y_0) = p_0 F_x(x_0, y_0, \bar{p}_0, \bar{q}_0) + q_0 F_y(x_0, y_0, \bar{p}_0, \bar{q}\,) \\ - \bar{p}_0 F_x(x_0, y_0, p_0, q_0) - \bar{q}_0 F_y(x_0, y_0, p_0, q_0), \end{aligned} \tag{8}$$

where

$$p_0 = \cos\vartheta_0, \quad q_0 = \sin\vartheta_0, \quad \bar{p}_0 = \cos\bar{\vartheta}_0, \quad \bar{q}_0 = \sin\bar{\vartheta}_0,$$

we have the result:

If the condition

$$\Omega(x_0, y_0) \neq 0 \tag{9}$$

is satisfied, there exists one and but one system of functions

$$t = t(a), \quad \bar{\vartheta} = \bar{\vartheta}(a), \tag{10}$$

of class C' in the vicinity of $a = a_0$, which satisfies the two equations (5) *and the initial conditions*

$$t(a_0) = t_0, \quad \bar{\vartheta}(a_0) = \bar{\vartheta}_0. \tag{11}$$

The functions (10) represent, at least for the vicinity of the point P_0, the solution of the problem proposed above.

From our assumption (2), applied to the point P_0 and the branch $P_0 P_2$, it follows that

$$F_1(\phi[t(a), a], \psi[t(a), a], \cos\vartheta(a), \sin\bar{\vartheta}(a)) \neq 0$$

for all sufficiently small values of $|a - a_0|$. Hence† it is possible to construct one and but one extremal

$$\bar{\mathfrak{E}}_a: \quad x = \bar{\phi}(t, a), \quad y = \bar{\psi}(t, a) \tag{12}$$

through the point P in the direction $\vartheta(a)$. The parameter t can be so selected that also on $\bar{\mathfrak{E}}_a$ the value $t = t(a)$ furnishes the point P, so that

$$\bar{\phi}[t(a), a] = \phi[t(a), a], \quad \bar{\psi}[t(a), a] = \psi[t(a), a]. \tag{13}$$

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* Compare *Lectures*, § 24, b) equations (8) and (10).

† According to Cauchy's existence theorem on differential equations; compare *Lectures*, § 25, b).

We thus obtain a broken extremal $\mathfrak{E}_a + \bar{\mathfrak{E}}_a$ with a corner at P, on which the parameter t varies continuously. If we let a vary, we obtain a *set of broken extremals.* We shall call the set (12) the *set of extremals complementary* to the set (3). On account of (11) it contains, for $a = a_0$, the extremal $\bar{\mathfrak{E}}_0$ of which the arc $P_0 P_2$ forms a part.

From the properties of the integrals of a system of differential equations as functions of their initial values,* it follows that the functions $\bar{\phi}(t, a)$, $\bar{\psi}(t, a)$ have the same continuity properties as the functions $\phi(t, a)$, $\psi(t, a)$, in a domain

$$t_0 - \bar{h}_0 \leqq t \leqq t_2 + h_2, \quad | a - a_0 | \leqq \bar{d}.$$

§2. *The Corner-Curve.*†

If we let a vary, the corner P describes a curve $\tilde{\mathfrak{C}}$, which we call the "*corner-curve*". If we define the functions $\tilde{x}(a)$, $\tilde{y}(a)$ by the equations

$$\tilde{x}(a) = \phi[t(a), a], \quad \tilde{y}(a) = \psi[t(a), a], \tag{14}$$

or, what amounts to the same thing according to (13),

$$\tilde{x}(a) = \bar{\phi}[t(a), a], \quad \tilde{y}(a) = \bar{\psi}[t(a), a], \tag{14a}$$

the corner-curve is given in parameter-representation by the equations

$$\tilde{\mathfrak{C}}: \quad x = \tilde{x}(a), \quad y = \tilde{y}(a),$$

and any particular value of a furnishes that point of $\tilde{\mathfrak{C}}$ which is the corner for the corresponding broken extremal $\mathfrak{E}_a + \bar{\mathfrak{E}}_a$.

We propose first to compute the slope $\tan \tilde{\vartheta}$ of the tangent to the corner-curve $\tilde{\mathfrak{C}}$ at the point P.

From the definition of the functions $\tilde{x}$, $\tilde{y}$, we obtain for their derivatives with respect to a:

$$\tilde{x}' = \phi_t t'(a) + \phi_a, \quad \tilde{y}' = \psi_t t'(a) + \psi_a;$$

and from (5) we obtain, according to the rules for the differentiation of implicit functions,

$$t'(a) = -\frac{J_{a\bar{\vartheta}}}{J_{t\bar{\vartheta}}},$$

where

$$J_{a\bar{\vartheta}} = \frac{\partial(\Phi, \Psi)}{\partial(a, \bar{\vartheta})}.$$

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* Compare Kneser, *Lehrbuch der Variationsrechnung*, §27, and Bliss, The Solution of Differential Equations of the First Order as Functions of their Initial Values, *Annals of Mathematics*, Ser. 2, Vol. VI, p. 49.

† Caratheodory's "Knickpunkt-Curve".

But

$$\Phi_a = F_{x'x}\phi_a + F_{x'y}\psi_a + F_{x'x'}\phi_{ta} + F_{x'y'}\psi_{ta} - \overline{F}_{x'x}\phi_a - \overline{F}_{x'y}\psi_a,$$
$$\Psi_a = F_{y'x}\phi_a + F_{y'y}\psi_a + F_{y'x'}\phi_{ta} + F_{y'y'}\psi_{ta} - \overline{F}_{y'x}\phi_a - \overline{F}_{y'y}\psi_a;$$

the functions $\overline{F}_{x'x}$, $\overline{F}_{x'y}$, $\overline{F}_{y'x}$, $\overline{F}_{y'y}$ are positively homogeneous of dimension zero with respect to their last two arguments $\bar{p}$, $\bar{q}$; hence we may replace $\bar{p}$ and $\bar{q}$ by $\phi_t(t, a)$ and $\bar{\psi}_t(t, a)$ respectively. This being done, we express all the partial derivatives of F in terms of Weierstrass' functions*: F_1, L, M, N. The result is

$$\begin{aligned} \Phi_a &= -A\phi_a - B\psi_a - y'\Delta_t F_1 - \bar{y}'\overline{F}_1(\phi_a\bar{y}'' - \psi_a\bar{x}''), \\ \Psi_a &= -B\phi_a - C\psi_a + x'\Delta_t F_1 + \bar{x}'\overline{F}_1(\phi_a\bar{y}'' - \psi_a\bar{x}''), \end{aligned} \tag{15}$$

where

$$\begin{aligned} &x' = \phi_t(t, a),\ y' = \psi_t(t, a);\ \bar{x}' = \bar{\phi}_t(t, a),\ \bar{y}' = \bar{\psi}_t(t, a); \\ &\bar{x}'' = \bar{\phi}_{tt}(t, a),\ \bar{y}'' = \bar{\psi}_{tt}(t, a);\ \Delta(t, a) = \phi_t\psi_a - \psi_t\phi_a, \\ &A = \overline{L} - L,\ B = \overline{M} - M,\ C = \overline{N} - N; \end{aligned}$$

the quantities L, M, N refer to the point P and the extremal $\mathfrak{E}_a$, the quantities $\overline{L}, \overline{M}, \overline{N}$ to the point P and the extremal $\overline{\mathfrak{E}}_a$. Finally, the last two arguments of F_1 and $\overline{F}_1$ are x', y' and $\bar{x}'$, $\bar{y}'$ respectively.

From (15) and (6) we obtain

$$J_{a\bar{\vartheta}} = (\bar{x}'^2 + \bar{y}'^2)\overline{F}_1\{\phi_a(A\bar{x}' + B\bar{y}') + \psi_a(B\bar{x}' + C\bar{y}') - \Delta_t F_1(x'\bar{y}' - y'\bar{x}')\}. \tag{16}$$

At the same time the expression (7) for $J_{t\bar{\vartheta}}$ may be thrown into another form, if we remember the homogeneity properties of F_1, F_x, F_y and make use of the relations†

$$Lx' + My' = F_x, \quad Mx' + Ny' = F_y;$$

we thus obtain

$$J_{t\bar{\vartheta}} = (\bar{x}'^2 + \bar{y}'^2)\overline{F}_1[Ax'\bar{x}' + B(x'\bar{y}' + y'\bar{x}') + Cy'\bar{y}']. \tag{17}$$

The comparison between the two expressions for $J_{t\bar{\vartheta}}$ leads to a second form for the quantity $\Omega(x, y)$; viz.,

$$\Omega(x, y) = Ap\bar{p} + B(p\bar{q} + q\bar{p}) + Cq\bar{q}. \tag{18}$$

We thus finally obtain

$$\begin{aligned} \tilde{x}' &= \frac{-\Delta(B\bar{x}' + C\bar{y}') + x'\Delta_t F_1(x'\bar{y}' - y'\bar{x}')}{Ax'\bar{x}' + B(x'\bar{y}' + y'\bar{x}') + Cy'\bar{y}'}, \\ \tilde{y}' &= \frac{\Delta(A\bar{x}' + B\bar{y}') + y'\Delta_t F_1(x'\bar{y}' - y'\bar{x}')}{Ax'\bar{x}' + B(x'\bar{y}' + y'\bar{x}') + Cy'\bar{y}'} \end{aligned} \tag{19}$$

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* Compare *Lectures*, Chap. IV, equations (11 a) and (35).

† Compare *Lectures*, p. 132.

Hence follows, *for the slope* $\tan \tilde{\mathfrak{S}}$ *of the tangent to the corner-curve* $\tilde{\mathfrak{C}}$ *at the point* P, the expression

$$\tan \tilde{\mathfrak{S}} = \frac{\Delta\,(A\,\bar{x}' + B\,\bar{y}') + y'\,\Delta_t\,F_1\,(x'\,\bar{y}' - y'\,\bar{x}')}{-\,\Delta\,(B\,\bar{x}' + C\,\bar{y}') + x'\,\Delta_t\,F_1\,(x'\,\bar{y}' - y'\,\bar{x}')}. \tag{20}$$

§ 3. *Interrelation Between the Slope of the Corner-Curve at* P_0 *and the Focal-Points of the Set of Broken Extremals.*

We now consider in particular the question how the slope $\tan \tilde{\mathfrak{S}}_0$ of the tangent to the corner-curve at P_0 depends upon the choice of the set of extremals (3). For this purpose we have to put $a = a_0$ in (20), and consequently, according to (11), the argument $t = t\,(a)$, in x', y'; $\bar{x}'$, $\bar{y}'$, $\Delta\,(t, a)$, etc., equal to t_0. In the resulting expression for $\tan \tilde{\mathfrak{S}}_0$, the Jacobian $\Delta\,(t_0, a_0)$ and its derivative $\Delta_t\,(t_0, a_0)$ are the only quantities which depend upon the choice of the set of extremals (3).

The function $\Delta\,(t, a_0)$, in its turn, is determined to a constant factor by the condition that it satisfies Jacobi's differential equation* for the extremal $\mathfrak{C}_0$, viz.,

$$F_2\,u - \frac{d}{dt}\left(F_1\,\frac{du}{dt}\right) = 0, \tag{21}$$

and by one of its zeros. Let $t = \tau$ be the zero of $\Delta\,(t, a_0)$ next smaller than t_0, so that the corresponding point of $\mathfrak{C}_0$, which we denote by Q, is the focal point† of the set (3) on $\mathfrak{C}_0$. Then

$$\Delta\,(t, a_0) = \text{Const.}\,\Theta\,(t, \tau),$$

where $\Theta\,(t, \tau)$ is the function which determines in Weierstrass'‡ theory the conjugate point to Q. We may therefore write

$$\tan \tilde{\mathfrak{S}}_0 = \frac{\alpha\,\Theta\,(t_0, \tau) + \beta\,\Theta_t\,(t_0, \tau)}{\gamma\,\Theta\,(t_0, \tau) + \delta\,\Theta_t\,(t_0, \tau)}, \tag{22}$$

where

$$\begin{aligned} \alpha &= A_0\,\bar{p}_0 + B_0\,\bar{q}_0, & \beta &= q_0\,F_1\,(t_0)\sin\,(\bar{\mathfrak{S}}_0 - \mathfrak{S}_0)\,(x_0'^2 + y_0'^2), \\ \gamma &= -\,(B_0\,\bar{p}_0 + C_0\,\bar{q}_0), & \delta &= p_0\,F_1\,(t_0)\sin\,(\bar{\mathfrak{S}}_0 - \mathfrak{S}_0)\,(x_0'^2 + y_0'^2), \end{aligned} \tag{23}$$

the subscript 0 indicating that the quantities to which it is affixed are to be computed for the point P_0.

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* Compare *Lectures*, pp. 40 and 200.

† Compare Kneser, *Lehrbuch der Variationsrechnung*, § 24, and my *Lectures*, § 38.

‡ Compare *Lectures*, p. 135.

The coefficients α, β, γ, δ are therefore independent of τ. Hence *the slope of the corner-curve* $\tilde{\mathfrak{C}}$ *at* P_0 *is the same for all sets of extremals* (3) *which have the same focal point* Q, the set of extremals through the point Q being included among the latter.

We examine next how the slope $\tan \tilde{\vartheta}_0$ varies when the focal point Q describes the extremal $\mathfrak{E}_0$. For this purpose, we compute the derivative of $\tan \tilde{\vartheta}_0$ with respect to τ:

$$\frac{d \tan \tilde{\vartheta}_0}{d\tau} = -\frac{(\alpha\delta - \beta\gamma)\,\{\Theta(t_0, \tau)\,\Theta_{t\tau}(t_0, \tau) - \Theta_t(t_0, \tau)\,\Theta_\tau(t_0, \tau)\}}{\{\gamma\,\Theta(t_0, \tau) + \delta\,\Theta_t(t_0, \tau)\}^2}.$$

But from the definition of $\Theta(t, \tau)$ it follows that

$$\begin{aligned}\Theta(t_0, \tau)\,\Theta_{t\tau}(t_0, \tau) - \Theta_t(t_0, \tau)\,\Theta_\tau(t_0, \tau)\\ = [\vartheta_1(t_0)\,\vartheta_2'(t_0) - \vartheta_2(t_0)\,\vartheta_1'(t_0)]\,[\vartheta_1(\tau)\,\vartheta_2'(\tau) - \vartheta_2(\tau)\,\vartheta_1'(\tau)],\end{aligned}$$

where $\vartheta_1(t)$, $\vartheta_2(t)$ are two linearly independent solutions of Jacobi's differential equation (21). Hence from the theory of linear differential equations it follows* that

$$\vartheta_1(t)\,\vartheta_2'(t) - \vartheta_2(t)\,\vartheta_1'(t) = \frac{k}{F_1(t)},$$

where k is a constant different from zero.

On the other hand we get, on substituting the values of α, β, γ, δ,

$$a\delta - \beta\gamma = F_1(t_0)\sin(\vartheta_0 - \tilde{\vartheta}_0)\,\Omega(x_0, y_0)\,(x_0'^2 + y_0'^2).$$

Hence it follows that

$$\frac{d}{d\tau}\tan \tilde{\vartheta}_0 = \frac{-k^2(x_0'^2 + y_0'^2)\sin(\vartheta_0 - \tilde{\vartheta}_0)\,\Omega(x_0, y_0)}{F_1(\tau)\,\{\gamma\,\Theta(t_0, \tau) + \delta\,\Theta_t(t_0, \tau)\}^2}. \tag{24}$$

We suppose for the further discussion that

$$\vartheta_0 - \tilde{\vartheta}_0 \not\equiv 0 \pmod{\pi}, \tag{25}$$

and that the inequality (2) holds not only for the arc $P_1 P_0$ of the extremal $\mathfrak{E}_0$ but also for its continuation beyond P_1, at least as far as the point P_0' $(t = t_0')$ whose conjugate the point P_0 is.

And now we let τ increase from t_0' to t_0; *i. e.*, we let the point Q describe the extremal $\mathfrak{E}_0$ from P_0' to P_0. The derivative of $\tan \tilde{\vartheta}_0$ will then always have a

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* Compare, for instance, *Lectures*, p. 58, footnote [2].

constant sign, since $\Omega(x_0, y_0)$, which is independent of τ, is supposed to be different from zero. For $\tau = t_0'$ and $\tau = t_0$, but for no other value between them, the function $\Theta(t_0, \tau)$ vanishes and $\tan \tilde{\vartheta}_0$ takes the value

$$\tan \tilde{\vartheta}_0 = \frac{\beta}{\delta} = \frac{q_0}{p_0} = \tan \vartheta_0.$$

Hence we have the result:

While the point Q describes the extremal $\mathfrak{E}_0$ from P_0' to P_0, the line $\tilde{\vartheta}_0$ revolves about the point P_0 from the initial position ϑ_0 constantly in the same sense through an angle of* 180°. The rotation takes place:

In positive sense, when $\Omega(x_0, y_0) \sin(\bar{\vartheta}_0 - \vartheta_0) < 0$;
In negative sense, when $\Omega(x_0, y_0) \sin(\bar{\vartheta}_0 - \vartheta_0) > 0$.

It passes therefore once and but once through the position $\bar{\vartheta}_0$. We denote the value of τ for which this takes place by e_0 and the corresponding point† of $\mathfrak{E}_0$ by E_0. For the discussion of sufficient conditions, it is important to distinguish whether the line $\tilde{\vartheta}_0$ lies in the angle‡ between the two branches $P_1 P_0$ and $P_0 P_2$ or outside of it. Four cases must be distinguished according to the signs of $\Omega(x_0, y_0)$ and $\sin(\bar{\vartheta}_0 - \vartheta_0)$. The result is:

While the point Q moves from P_0' to E_0, the line $\tilde{\vartheta}_0$ revolves from the position ϑ_0 into the position $\bar{\vartheta}_0$, inside of the angle between $P_1 P_0$ and $P_0 P_2$ when $\Omega(x_0, y_0) > 0$, outside of it when $\Omega(x_0, y_0) < 0$. As the point Q moves on from E_0 to P_0, the line $\tilde{\vartheta}_0$ continues its rotation from the position $\bar{\vartheta}_0$ into the position ϑ_0, outside of the angle in question when $\Omega(x_0, y_0) > 0$, inside of it when $\Omega(x_0, y_0) < 0$.

Conversely: To every line $\tilde{\vartheta}_0$ through the point P_0 which is tangent to neither of the two arcs $P_1 P_0$, $P_0 P_2$ at P_0, there belongs one and but one point Q, between P_0' and P_0, such that the corner-curve for every set of extremals (3) for which Q is the focal point, touches the line $\tilde{\vartheta}_0$ at P_0.

The value of τ belonging to a given line $\tilde{\vartheta}_0$ is obtained by solving equation (22) with respect to τ. The equation may be thrown into the form

$$\begin{aligned}&[A_0 \bar{p}_0 \tilde{p}_0 + B_0(\bar{p}_0 \tilde{q}_0 + \bar{q}_0 \tilde{p}_0) + C_0 \bar{q}_0 \tilde{q}_0] \,\Theta(t_0, \tau)\\ &\qquad - (x_0'^2 + y_0'^2) F_1(t_0) \sin(\bar{\vartheta}_0 - \vartheta_0) \sin(\tilde{\vartheta}_0 - \vartheta_0)\, \Theta_t(t_0', \tau) = 0,\end{aligned} \tag{26}$$

where

$$\tilde{p}_0 = \cos \tilde{\vartheta}_0, \quad \tilde{q}_0 = \sin \tilde{\vartheta}_0.$$

* *I. e.*, the line through P_0 of slope $\tan \tilde{\vartheta}_0$.

† Caratheodory denotes this point by E_1; see *Dissertation*, p. 31.

‡ *I. e.*, that one of the two angles formed by the half-rays $\bar{\vartheta}_0$ and $\vartheta_0 + \pi$ which is less than π.

Case I: $\sin(\bar{\vartheta}_0 - \vartheta_0) > 0$, $\Omega(x_0, y_0) > 0$.

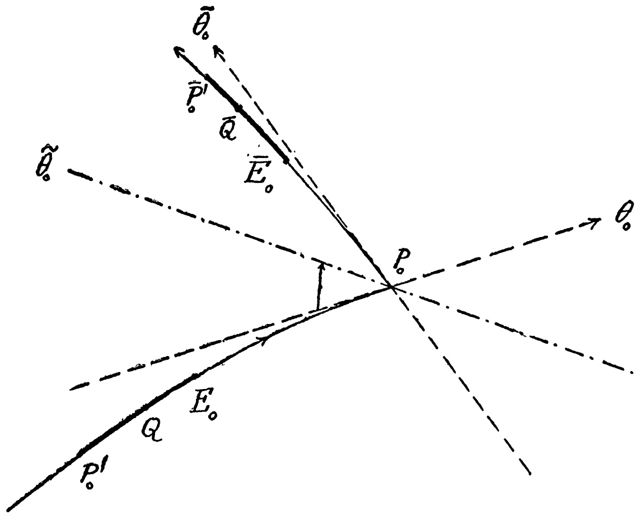

FIG. 2.

Case II: $\sin(\bar{\vartheta}_0 - \vartheta_0) > 0$, $\Omega(x_0, y_0) < 0$.

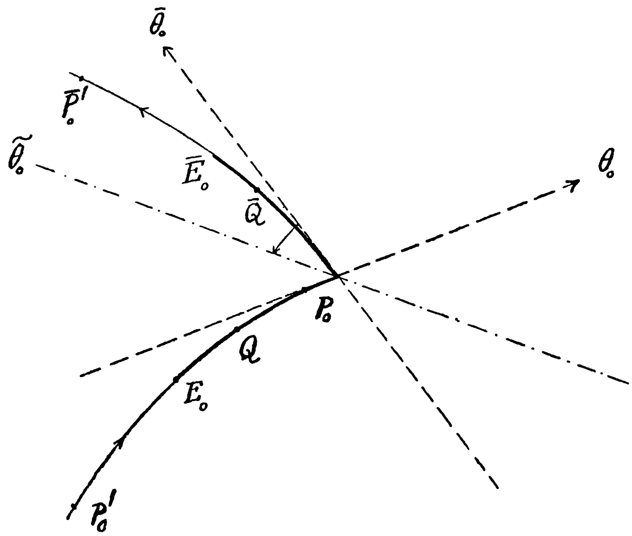

FIG. 3.

Case III: $\sin(\bar{\vartheta}_0 - \vartheta_0) < 0,\ \Omega(x_0, y_0) > 0.$

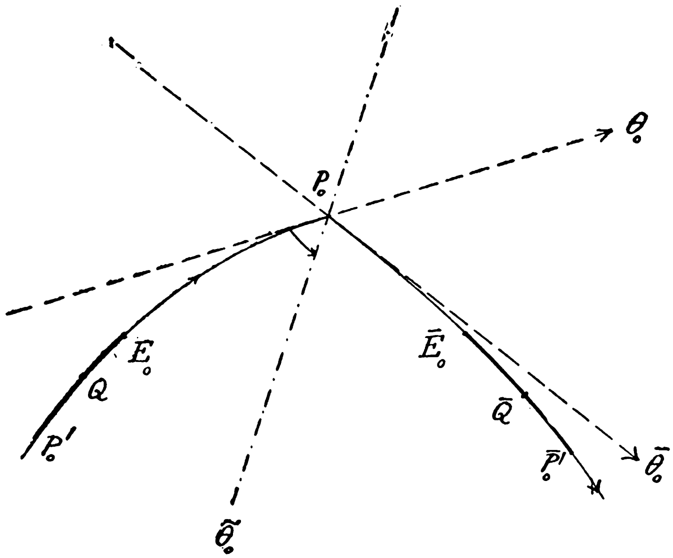

Fig. 4.

Case IV: $\sin(\bar{\vartheta}_0 - \vartheta_0) < 0,\ \Omega(x_0, y_0) < 0.$

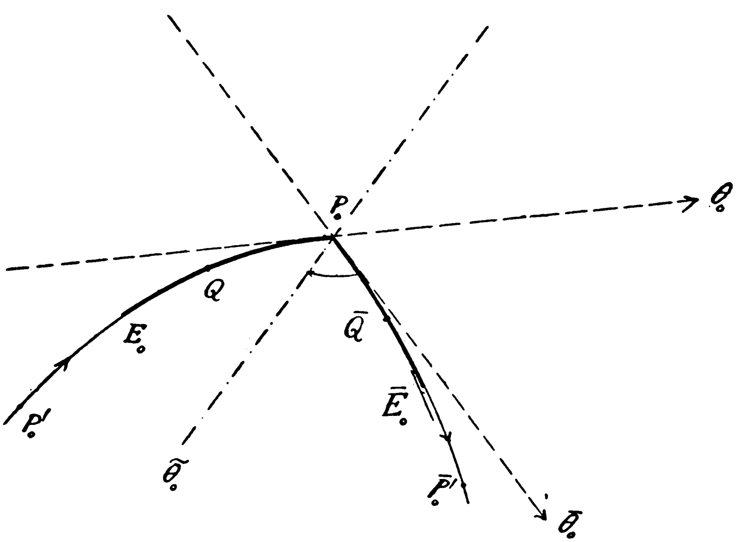

Fig. 5.

In particular, the equation for the determination of the parameter e_0 of the point E_0 is obtained by putting, in (26), $\tilde{\mathfrak{S}}_0 = \mathfrak{S}_0$.

§ 4. *The Conjugate Points of Discontinuous Solutions.*

Let, for a moment, the equations (12) represent any set of extremals containing, for $a = a_0$, the extremal $\bar{\mathfrak{E}}_0$. We may then propose for the set (12) the same problem which we have solved in § 1 for the set (3). The only difference will be that in the equations (5) the symbols ϕ, ψ, $\tilde{\mathfrak{S}}$ must be interchanged with $\bar{\phi}$, $\bar{\psi}$, $\mathfrak{S}$, and the same interchange must be applied in the results; in this process the quantities A, B, C are changed into $-A$, $-B$, $-C$. Accordingly, if $\bar{Q}\,(t = \bar{\tau})$ be the focal point of the set (12) on $\bar{\mathfrak{E}}_0$, the slope of the corner-curve belonging to the set (12) at P_0 is

$$\tan \bar{\tilde{\mathfrak{S}}}_0 = \frac{\bar{\alpha}\,\bar{\Theta}\,(t_0, \bar{\tau}) + \bar{\beta}\,\bar{\Theta}_t\,(t_0, \bar{\tau})}{\bar{\gamma}\,\bar{\Theta}\,(t_0, \bar{\tau}) + \bar{\delta}\,\bar{\Theta}_t\,(t_0, \bar{\tau})}, \tag{27}$$

where the quantities $\bar{\alpha}, \bar{\beta}, \bar{\gamma}, \bar{\delta}$ are derived from $\alpha, \beta, \gamma, \delta$ by the above interchange, and $\bar{\Theta}$ has the same meaning for $\bar{\mathfrak{E}}_0$ as Θ for $\mathfrak{E}_0$.

Conversely, we obtain the value of $\bar{\tau}$ corresponding to a given line $\bar{\tilde{\mathfrak{S}}}$ by solving equation (27). We denote the value of $\bar{\tau}$ corresponding to the particular line $\mathfrak{S}_0$ by $\bar{e}_0$ and the corresponding point of $\bar{\mathfrak{E}}_0$ by $\bar{E}_0$; this point lies between the point P_0 and its conjugate $\bar{P}_0'\,(t = \bar{t}_0')$ on $\bar{\mathfrak{E}}_0$.

Let now the equation (12) denote again, as in § 1, the particular set of extremals complementary to the set (3). The two sets (3) and (12) will then have the corner-curve in common; hence we have, in this case,

$$\bar{\tilde{\mathfrak{S}}}_0 = \tilde{\mathfrak{S}}_0 .$$

We obtain, therefore, the focal point of the set (12) complementary to the set (3) by equating the right-hand members of the two equations (22) and (27) and solving the equation thus obtained with respect to τ. After some reductions the following result is obtained:

If $t = \tau$ be the parameter of the focal point Q of the set of extremals (3) *on $\mathfrak{E}_0$, and $t = \bar{\tau}$ the parameter of the focal point $\bar{Q}$ of the set* (12), *complementary to* (3), *on $\bar{\mathfrak{E}}_0$, then the following relation holds between τ and $\bar{\tau}$:*

$$\left.\begin{aligned}
&(A_0 C_0 - B_0^2)\,\Theta(t_0, \tau)\,\overline{\Theta}(t_0, \bar{\tau}) \\
&\quad - (x_0'^2 + y_0'^2)\,F_1(t_0)\,(A_0 p_0^2 + 2\,B_0 p_0 q_0 + C_0 q_0^2)\,\frac{\partial\,\Theta(t_0, \tau)}{\partial\,t_0}\,\overline{\Theta}(t_0, \tau) \\
&\quad + (\bar{x}_0'^2 + \bar{y}_0'^2)\,\overline{F}_1(t_0)\,(A_0 \bar{p}_0^2 + 2\,B_0 \bar{p}_0 \bar{q}_0 + C_0 \bar{q}_0^2)\,\Theta(t_0, \tau)\,\frac{\partial\,\overline{\Theta}}{\partial\,t_0}(t_0, \tau) \\
&- (x_0'^2 + y_0'^2)\,(\bar{x}_0'^2 + \bar{y}_0'^2)\,F_1(t_0)\,\overline{F}_1(t_0)\,\sin^2(\bar{\vartheta}_0 - \vartheta_0)\,\frac{\partial\,\Theta(t_0, \tau)}{\partial\,t_0}\,\frac{\partial\,\overline{\Theta}(t_0, \bar{\tau})}{\partial\,t_0} = 0.
\end{aligned}\right\} \quad (28)$$

The two points Q and $\overline{Q}$ are called, according to Caratheodory,* *a pair of conjugate points of the broken extremal* $\mathfrak{E}_0 + \overline{\mathfrak{E}}_0$. According to a previous remark, the point $\overline{Q}$ conjugate to Q on $\mathfrak{E}_0 + \overline{\mathfrak{E}}_0$ may also be defined as *the focal point on* $\overline{\mathfrak{E}}_0$ *of the set of extremals which is complementary to the set of extremals through the point* Q.

In Figs. 2 to 5 the interrelation between the points Q and $\overline{Q}$ and the line $\tilde{\vartheta}_0$ is indicated. For instance, in Case I the point $\overline{Q}$ moves on $\overline{\mathfrak{E}}_0$ from $\overline{E}_0$ to $\overline{P}_0'$ while the point Q moves on $\mathfrak{E}_0$ from P_0' to E_0; at the same time the line $\tilde{\vartheta}_0$ revolves about P_0 from the position ϑ_0, in the sense of the arrow, into the position $\bar{\vartheta}_0$.

The conjugate points thus defined play for the discontinuous solutions a rôle similar to that of the ordinary conjugate points for continuous solutions, at least in the case when the line $\tilde{\vartheta}_0$ lies inside the angle of the two branches $P_1 P_0$, $P_0 P_2$. We refer in this respect to Caratheodory's dissertation, § 9.

THE UNIVERSITY OF CHICAGO, *January* 29, 1907.

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* Caratheodory restricts, however, the definition to the case when the line $\tilde{\vartheta}_0$ lies inside of the angle of the two branches $P_1 P_0$, $P_0 P_2$.